The Tutte polynomial and the automorphism group of a graph

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Abstract

A graph G is said to be p-periodic, if the automorphism group $Aut(G)$ contains an element of order p which preserves no edges. In this paper, we investigate the behavior of graph polynomials (Negmai and Tutte) with respect to graph periodicity. In particular, we prove that if p is a prime, then the coefficients of the Tutte polynomial of such a graph satisfy a certain necessary condition. This result is illustrated by an example where the Tutte polynomial is used to rule out the periodicity of the Frucht graph.

Key words. Automorphism group, graph periodicity, Tutte and Negami polynomials.
MSC. 05C31.

1 Introduction

Throughout this paper, a graph is the geometric realization of a 1-dimensional CW-complex. Let G be a graph with edge set $E(G)$ and vertex set $V(G)$. Let $p \geq 2$ be an integer, the graph G is said to be *p-periodic* if its automorphism group contains an element h of order p subject to the condition that no edge of G is fixed under the action of $\mathbb{Z}_p =< h >$. Equivalently, the finite cyclic group $\mathbb{Z}_p$ acts on the graph preserving the incidence and such that the restriction of this action to $E(G)$ is fixed point free. Let G be a $p-$periodic graph. If we identify the

vertices which belong to the same orbit to a single vertex and the edges which belong to the same orbit to a single edge, then we obtain the quotient graph of G under the action of $< h >$. This quotient graph is denoted hereafter by $\overline{G}$.

The chromatic polynomial $P_G(\lambda)$ is a classical tool in graph theory, which counts the number of proper colorings of the vertices of the graph with λ distinct colors. This polynomial can be recursively defined using a simple contraction-deletion formula. The Tutte polynomial $\tau_G(x, y)$ is an isomorphism invariant of graphs. It is a two-variable polynomial with integral coefficients which specializes to the chromatic polynomial. The Tutte polynomial can be also defined through a contraction-deletion formula as explained in Section 2. In this paper, we consider a modified version of the Tutte polynomial by setting $T_G(s, t) = \tau_G(s+1, t+1)$. Then $T_G(s, t) = \sum_{i,j} a_{i,j} s^i t^j$ where $a_{i,j}$ are integers. The importance of the Tutte polynomial comes not only from the many information it carries about the graph, but also from its connection to other fields such as knot theory and statistical physics. Actually, it is well known that the Tutte polynomial specializes to the partition function of the $q-$state Potts model [11].

The Tutte polynomial has been generalized into several directions. For instance, Negami [9] introduced a three variable polynomial $N_G(u, x, y)$ which specializes to the Tutte polynomial (see Section 2). Another interesting generalization has been obtained by Murasugi [6] who defined a polynomial invariant of weighted graphs, refereed to here as Murasugi polynomial. Bollobas and Riordan [1] introduced a kind of universal Tutte polynomial of colored graphs with respect to the deletion-contraction formula.

The purpose of this paper is to investigate the relationship between the automorphism group of a given graph and its polynomials. More precisely, we study the behavior of the Negami polynomial of periodic graphs asking whether this polynomial reflects the periodicity of graphs. One of the main motivations of this study is the nice behavior of the coefficients of the HOMPLYPT polynomial of symmetric links introduced in [2] and [3] for instance. The main results in this paper confirm that the coefficients of the Negami and Tutte polynomials strongly reflect

the periodicity of graphs.

Theorem 1.1. *Let p be a prime and assume that G is a connected p-periodic graph. Then $N_G(u,x,y) \cong \sum_{i,j} u^i x^{q-jp} y^{jp}$ modulo p, where q is the number of edges of G.*

Corollary 1.2. *Let p be a prime and assume that G is a connected p-periodic graph. Then $T(s,t) = \sum_{i,j} a_{i,j} s^i t^j$ where $a_{i,j} \cong 0$ modulo p whenever $j - i \ncong 1 - r$ modulo p, where r is the number of vertices of G.*

Corollary 1.3. *Let G be a self-dual connected planar graph and p be a prime. If G is $p-$periodic, then $r \cong 1$ modulo p, where r is the number of vertices of G.*

Example 1.4. Let P be the Petersen graph in Figure 1. It is well known that this graph is 5-periodic. The modified Tutte polynomial modulo 5 is: $T_P(s,t) = s^4 + s^9 + 2t + 2s^5t + st^2 + t^6$. Since the only nonzero coefficients are $a_{4,0}, a_{9,0}, a_{0,1}, a_{5,1}, a_{1,2}, a_{0,6}$, the necessary condition given by the Theorem is satisfied.

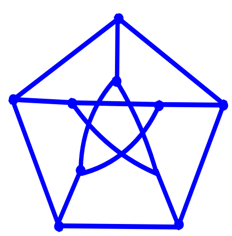

FIGURE 1

Application 1.5. Corollary 1.2 can be applied to rule out the possibility of being periodic for certain graphs as it is illustrated in the following example. Let F be the Frucht graph in Figure 2. The reduced modulo 3, Tutte polynomial of F, is given by

$T_F(s,t) = 1 + s + s^2 + 2s^3 + s^6 + 2s^7 + s^{11} + st + s^2t + 2s^3t + 2s^4t + s^5t + 2s^6t + 2s^7t + s^8t + 2t^2 + s^2t^2 + s^4t^2 + 2s^5t^2 + 2s^6t^2 + s^7t^2 + t^3 + 2st^3 + s^2t^3 + s^4t^3 + s^5t^3 + 2t^4 + 2st^4 + 2s^2t^4 + 2s^3t^4 + st^5 + t^7$.

Since $T_F(s,t)$ does not satisfy the periodicity condition given by Corollary 1.2. Then the Frucht graph is not $3-$periodic.

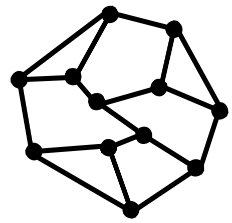

FIGURE 2

2 Negami's polynomial

The Negami polynomial is a three-variable isomorphism invariant of graphs [9]. It can be defined by a simple expansion formula. Let G be a graph with edges set $E(G)$. Assume that G has r vertices, q edges and $w(G)$ components. Then the Negami polynomial is defined by the following formula:

$$N_G(u,x,y) = \sum_{Y \subset E(G)} u^{w(G-Y)} x^{q-|Y|} y^{|Y|}.$$

According to [9], the Tutte polynomial is obtained as a specialization of the Negami polynomial:

$$\tau_G(x,y) = (y-1)^{-r}(x-1)^{-w(G)} N((x-1)(y-1), 1, y-1).$$

Thus we have

$$T_G(s,t) = \tau(s+1, t+1) = t^{-r} s^{-w(G)} N(st, 1, t).$$

If G is a graph and e is an edge, we denote by $G-e$ the graph obtained from G by deleting the edge e and by G/e we denote the graph obtained by contracting the edge e. The Tutte polynomial can be defined using the following deletion-contraction formula:

$$\tau_G(x,y) = \begin{cases} x\tau_{G/e}(x,y) & \text{if } e \text{ is a bridge} \\ y\tau_{G-e}(x,y) & \text{if } e \text{ is a loop} \\ \tau_{G-e}(x,y) + \tau_{G/e}(x,y) & \text{if } e \text{ is an ordinary edge} \end{cases}$$

Together with the condition $T(E_n) = 1$, where E_n is the graph with n vertices and no edges.

Proof of Theorem 1.1. Let G be a $p-$periodic connected graph with r vertices and q edges. The cyclic group $\mathbb{Z}_p$ acts on $\mathcal{P}(E(G))$. Since p is a prime then there are two kinds of orbits. Orbits which are made up of only one element and orbits which are made up of exactly p elements.

It is clear that if Y belongs to an orbit made up of p elements, then the contribution of these elements sums to zero modulo p in the expansion formula of $N_G(u,x,y)$. Consequently, in the computation of $N_G(u,x,y)$ modulo p, we shall consider only subsets Y which are fixed by the action of $\mathbb{Z}_p$. Since the action has no fixed edges, then the number of elements of $|Y|$ is an multiple of p, say jp. A priori, we don't have any control of the number of components $w(G-Y)$. Thus, the the polynomial is written on the form:

$$N_G(u,x,y) = \sum_{Y \subset E(G)} u^{w(G-Y)} x^{q-|Y|} y^{|Y|} \cong \sum_{i,j \in S} u^i x^{q-jp} y^{jp} \text{ modulo } p,$$

where S is a certain subset of $\mathbb{N} \times \mathbb{N}$. □

Proof of Corollary 1.2. Since G is a connected graph, then $T_G(s,t) = t^{-r}s^{-1}N_G(st,1,t)$. The congruence condition in Theorem 1.1 implies that modulo p, we have $T_G(s,t) \cong t^{-r}s^{-1}\sum_{i,j}(st)^i t^{jp}$. Hence, we get $T_G(s,t) \sum_{i,j} s^{i-1}t^{jp-r+i}$. This ends the proof of Corollary 1.2. □

Proof of Corollary 1.3. Let G^* be the dual graph of G. It is well known that $T_G(s,t) = T_{G^*}(t,s)$. Thus, if G is self-dual then $T_G(s,t) = T_G(t,s)$. If we assume that G is $p-$periodic then $T(s,t) = \sum_{i,j} a_{i,j}s^i t^j$ where

$a_{i,j} \cong 0$ modulo p whenever $j - i \ncong 1 - r$ modulo p and

$a_{i,j} \cong 0$ modulo p whenever $i - j \ncong 1 - r$ modulo p.

This implies that $a_{i,j} \cong 0$ modulo p whenever $r \ncong 1$ modulo p. □

3 Negami's polynomial and Murasugi congruence

In Knot theory, Murasugi's congruence refers to the relationship between the invariant of a periodic link and the invariant of its quotient link. Murasugi proved his congruence for the

Alexander polynomial of periodic knots [7], then extended it to the Jones polynomial [6] of periodic links. This congruence has been generalized to other link invariants and to the Yamada invariant of spatial graphs, [4, 5]. In [6], Murasugi introduced a new polynomial of weighted graphs and proved that this polynomial satisfies a certain congruence of Murasugi type. In this paragraph, we prove that this condition extends to the Negami and Tutte polynomials as well.

Theorem 3.1. *Let p be a prime and G be a connected $p-$periodic graph. Then $N_G(u,x,y) \cong (N_{\bar{G}}(u,x,y))^p$ modulo the ideal generated by p and $u^p - u$.*

Corollary 3.2. *Let p be a prime and G be a connected $p-$periodic graph. Then, $T_G(s,t) \cong (T_{\bar{G}}(s,t))^p$ modulo the ideal generated by p, $s^p - s$ and $t^p - t$.*

Proofs. A subset $Y \in \mathcal{P}(E(G))$ is called periodic if and only if Y is invariant by the $\mathbb{Z}_p$-action on $E(G)$. As we have seen in the proof of Theorem 1.1, only periodic subsets $Y \in \mathcal{P}(E(G))$ are to be considered in the computation of $N_G(u,x,y)$ modulo p. It can be easily seen that a periodic subset Y defines a subset $\overline{Y}$ of $E(\overline{G})$ and vise-versa. Moreover, we have $|Y| = p|\overline{Y}|$ and $q = p\overline{q}$ where $\overline{q}$ is the number of edges of the quotient graph $\overline{G}$.

On the other hand, it can be easily seen that if $\overline{G} - \overline{Y}$ has one component, then $G - Y$ can have one or p components. Hence, $u^{w(G-Y)}$ and $u^{w(\overline{G}-\overline{Y})}$ coincide modulo $u^p - u$. Now, modulo p and $u^p - u$, the Negami polynomial of G can be computed as follows:

$$N_G(u,x,y) \cong \sum_{Y \subset E(G)} u^{w(G-Y)} x^{q-|Y|} y^{|Y|} \cong \sum_{\overline{Y} \subset E(\overline{G})} u^{w(\overline{G}-\overline{Y})} x^{p\overline{q}-p|\overline{Y}|} y^{p|\overline{Y}|} \cong [N_{\overline{G}}(u,x,y)]^p.$$

The proof of Corollary 3.2 is straightforward. □

Remark. Since the periodicity criterion introduced in Theorem 3.1 involves both the polynomials of the graph and its quotient. Then it seems difficult to find an example where this condition is used to rule out the periodicity of the graph. However, this might be possible if we add some extra conditions on the graph or its quotient as in the following case. Assume that G is a p-periodic graph having a cycle of order p which is invariant by the action of $\mathbb{Z}_p$. Let us first write the criterion of Theorem 3.1 for the special case of the chromatic polynomial.

According to [9], we have $P_G(\lambda) = N_G(\lambda, -1, 1)$. Then Theorem 3.1 implies

$$P_G(\lambda) \cong (P_{\overline{G}}(\lambda))^p \quad \text{modulo the ideal generated by } p \text{ and } \lambda^p - \lambda.$$

If G has a cycle of length p which is invariant under the action of $\mathbb{Z}_p$ then the quotient graph has a loop and thus $P_{\bar{G}}(\lambda) = 0$. Hence:

$$P_G(\lambda) \cong 0 \quad \text{modulo the ideal generated by } p \text{ and } \lambda^p - \lambda.$$

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